Reliability

It is good practice to report a pt. estimate (e.g. posterior mean, posterior mode), together w/ a measure of reliability e.g. CI, HPD region, posterior variance, Laplace approx).

Confidence Intervals

A Frequentist CI is a probability statement about the interval

$$p(\underbrace{l(Y)}_{\text{random}} < \theta < \underbrace{u(Y)}_{\text{random}} \mid \theta) = 1-\alpha$$

A Bayesian CI is a probability statement about θ, i.e.

$$p(l(y) < \theta < u(y) \mid Y=y) = 1-\alpha$$

obs. data

Example: beta-binomial

let $y = y_1, \dots, y_n$

Likelihood: $p(y \mid \theta) = f(\theta) \underset{\theta}{\propto} \theta^{\Sigma y_i}(1-\theta)^{n-\Sigma y_i}$

Prior: $p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$

posterior (θ) $\propto$ likelihood · prior

$\propto \theta^{\boxed{a+\Sigma y_i}-1}(1-\theta)^{\boxed{b+n-\Sigma y_i}-1}$

$$\theta \mid y \sim \text{beta}(\underbrace{a+\Sigma y_i}_{\alpha_n}, \underbrace{b+n-\Sigma y_i}_{\beta_n})$$

95% Bayesian CI: qbeta(c(0.025, 0.975), α_n, β_n)

<u>HPD Region</u>: $\int_\theta p(\theta|y) \cdot \mathbb{1}_{\{p(\theta|y) \geq c\}} \, d\theta = 1-\alpha$

Find c, then set $\{\theta\}$ s.t. $p(\theta|y) \geq c$.

Laplace Approximation

<u>idea</u>: fit a Gaussian to mode of posterior.

<u>Method</u>: Taylor expand $\log p(\theta|y)$ about $\hat{\theta}_{MAP}$.

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}} \; p(\theta|y)$$

$\hat{\theta}_{MAP}$: $\frac{d}{d\theta} \log p(\theta|y) = 0$ and

locally concave, $\frac{d^2}{d\theta^2} \log p(\theta|y) \Big|_{\hat{\theta} \pm \varepsilon} < 0$

<u>Define</u>: $\log p(\theta|y) = L(\theta)$ for convenience

$$L(\theta) \approx L(\hat{\theta}) + \underbrace{\cancel{L'(\hat{\theta})}}_{0}(\theta - \hat{\theta}) + \frac{L''(\hat{\theta})(\theta - \hat{\theta})^2}{2}$$

$$p(\theta|y) = \exp\{L(\theta)\} = \underbrace{e^{L(\hat{\theta})} \cdot e^{\frac{1}{2}L''(\hat{\theta})(\theta-\hat{\theta})^2}}_{\substack{\text{kernel of a normal} \\ \text{w/ mean} = \hat{\theta} \\ \text{\& variance} = -1/L''(\hat{\theta})}}$$

$$\boxed{\theta | y \approx N\left(\hat{\theta}_{MAP}, \; -1/L''(\hat{\theta}_{MAP})\right)}$$

Example beta-binomial

$$p(\theta \mid y) = c \cdot \theta^{\alpha_n - 1}(1-\theta)^{\beta_n - 1}$$

$$L(\theta) = \log p(\theta \mid y) = \log c + (\alpha_n - 1)\log\theta + (\beta_n - 1)\log(1-\theta)$$

$$L'(\theta) = \frac{\alpha_n - 1}{\theta} - \frac{\beta_n - 1}{1-\theta}$$

$$\boxed{L''(\theta) = \frac{-(\alpha_n - 1)}{\theta^2} - \frac{(\beta_n - 1)}{(1-\theta)^2}}$$

set $L'(\theta) = 0$, then $\boxed{\hat{\theta}_{MAP} = \frac{\alpha_n - 1}{\beta_n + \alpha_n - 2}}$

$$p(\theta \mid y) \approx N(\hat{\theta}_{MAP}, -1/L''(\theta_{MAP}))$$