

"Completing the square"

①

Let $Y_i | \theta, \sigma^2 \stackrel{iid}{\sim} \text{normal}(\theta, \sigma^2)$

Let $\theta | \sigma^2 \sim \text{normal}(\mu_0, \tau_0^2)$

We want to prove that the full conditional posterior $\theta | \sigma^2, \underline{y}$ is normal with some posterior mean μ_n and posterior variance τ_n^2 . i.e. we want to prove:

$$\boxed{\theta | \sigma^2, \underline{y} \sim \text{normal}(\mu_n, \tau_n^2)}$$

PROOF:

STEP 1: Bayes' thm:

$$\begin{aligned} \underline{p(\theta | \sigma^2, \underline{y})} &\propto \underbrace{p(\underline{y} | \theta, \sigma^2)}_{\prod_{i=1}^n d\text{norm}(y_i, \theta, \sigma)} p(\theta | \sigma^2) \\ &\propto \exp\left\{-\frac{1}{2\sigma^2} \sum (y_i - \theta)^2\right\} \cdot \exp\left\{-\frac{1}{2\tau_0^2} (\theta - \mu_0)^2\right\} \end{aligned}$$

STEP 2: simplify by expanding quadratic terms and absorbing what we can into the normalizing constant. *Note: $\sum y_i = n\bar{y}$

$$\propto \exp\left\{-\frac{1}{2\sigma^2} (n\theta^2 - 2n\bar{y}\theta) - \frac{1}{2\tau_0^2} (\theta^2 - 2\mu_0\theta)\right\}$$

combine like terms:

$$\propto \exp\left\{-\frac{1}{2} \underbrace{\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)}_{\text{"a"}} \theta^2 + \underbrace{\left(\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)}_{\text{"b"}} \theta\right\}$$

define "a" & "b":

$$\propto \exp\left\{-\frac{1}{2} a \theta^2 + b \theta\right\}$$

(*) Notice: this is the kernel of a normal with mean = $\frac{b}{a}$
and variance = a^{-1}

therefore, $\theta | \sigma^2, \underline{y} \sim \text{normal}(\mu_n, \tau_n^2)$ where

$$\mu_n = \frac{b}{a} \quad \text{and} \quad \tau_n^2 = a^{-1}$$

and a, b are defined above. □

Proof of (*) on previous page: This answers the question: (2)
How did I recognize $\exp\left\{-\frac{1}{2}a\theta^2 + b\theta\right\}$ as the
kernel of a normal w/ mean $\frac{b}{a}$ and variance a^{-1} ?

Assume $\theta | y, \sigma^2 \sim \text{normal}\left(\frac{b}{a}, a^{-1}\right)$

Then

$$\begin{aligned} p(\theta | y, \sigma^2) &\propto \exp\left\{-\frac{1}{2}a\left(\theta - \frac{b}{a}\right)^2\right\} \\ &\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta - \frac{1}{2}a\frac{b^2}{a^2}\right\} \\ &\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta\right\} \underbrace{\left\{-\frac{1}{2}a\frac{b^2}{a^2}\right\}}_{\text{constant in } \theta} \end{aligned}$$

* Today's punchline:

Assume:

① $Y_i | \theta, \sigma^2 \sim N(\theta, \sigma^2)$

$$p(\theta, \sigma^2) = p(\theta | \sigma^2) p(\sigma^2)$$

② $\theta | \sigma^2 \sim N(\mu_0, \sigma^2 / \kappa_0)$

③ $1/\sigma^2 \sim \text{gamma}(\frac{\nu_0}{2}, \frac{\nu_0}{2} \sigma_0^2)$

then the posterior

$$p(\theta, \sigma^2 | y_1, \dots, y_n)$$

"marginal posterior"

$$= p(\theta | \sigma^2, y_1, \dots, y_n) p(\sigma^2 | y_1, \dots, y_n)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$
$$d\text{norm}(\theta; \mu_n, \tau_n) \text{dirgamma}(\sigma^2; \frac{\nu_n}{2}, \frac{\nu_n}{2} \sigma_n^2)$$

"full cond'l posterior of θ "

Today's agenda:

(1) sketch proof for $p(\sigma^2 | y_1, \dots, y_n)$

(2) sample from ^{joint} posterior

(3) sample from posterior predictive $p(\tilde{y} | y_1, \dots, y_n)$

↪ (time permitting)

Interpretation:

μ_0 : prior guess for θ

σ_0^2 : prior guess for σ^2

κ_0 : prior sample size for θ

ν_0 : prior sample size for σ^2

Assume

- ① $y_i | \theta, \sigma^2 \stackrel{iid}{\sim} N(\theta, \sigma^2)$
- ② $\theta | \sigma^2 \sim N(\mu_0, \sigma^2 / \kappa_0)$
- ③ $1/\sigma^2 \sim \text{gamma}(\nu_0/2, \frac{\nu_0 \sigma_0^2}{2})$

then the joint posterior

$$p(\theta, \sigma^2 | y) = \underbrace{p(\theta | \sigma^2, y)}_{\substack{\downarrow \\ \text{dnorm} \\ \checkmark \text{shown} \\ \text{previously}}} p(\sigma^2 | y)_{\substack{\downarrow \\ \text{dirgamma}}}$$

Goal: to prove $\sigma^2 | y \sim \text{invgamma}$

Sketch proof:

$$\begin{aligned} p(\sigma^2 | y) &\propto p(y | \sigma^2) p(\sigma^2) \\ &\propto p(\sigma^2) \int p(y, \theta | \sigma^2) d\theta \\ &\propto p(\sigma^2) \int \underbrace{p(y | \theta, \sigma^2)}_{\substack{\downarrow \\ \text{③}}} \underbrace{p(\theta | \sigma^2)}_{\substack{\downarrow \\ \text{①}}} d\theta \quad \text{②} \end{aligned}$$

Known by assumption:

The integral (†) above reduces to integrating a normal density.

Hint: collect (but do not absorb into normalizing constant) terms w/o θ outside integral.

Explicit bookkeeping:

(1) collect (but do not absorb into normalizing const!) terms w/o θ outside integral:

$$\left(2\pi\sigma^2\right)^{-n/2} \cdot \left(2\pi\sigma^2/K_0\right)^{-1/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}(\sum y_i^2) - \frac{K_0}{2\sigma^2}(\mu_0)^2\right\}$$

→ call this " $c_1(\sigma^2)$ "

$$\cdot \int \exp\left\{-\frac{1}{2\sigma^2}(-2(\sum y_i)\theta + n\theta^2) - \frac{K_0}{2\sigma^2}(-2\mu_0\theta + \theta^2)\right\} d\theta$$

(2) Compute the integral above using the kernel trick:
 hint: it's the kernel of a normal, you must complete the square.

$$\int e^{-\frac{1}{2}a\theta^2 + b\theta} d\theta = \left(2\pi/a\right)^{1/2} \exp\left\{\frac{1}{2}b^2/a\right\}$$

→ call this " $c_2(\sigma^2)$ "

where $a = \left(\frac{n}{\sigma^2} + \frac{K_0}{\sigma^2}\right)$

$b = \frac{(n\bar{y} + K_0\mu_0)}{\sigma^2}$

(3) Return to (1) and write down:

$$p(\sigma^2|y) \propto p(\sigma^2) \cdot c_1(\sigma^2) \cdot c_2(\sigma^2)$$