M-H acceptance ratio : r

$$r = \underbrace{\frac{\pi(\underline{\theta}^*)}{\pi(\underline{\theta}^{(s)})}}_{\text{ratio of posteriors}} \cdot \underbrace{\frac{J(\theta_1^{(s)} \mid \theta_1^*)}{J(\theta_1^* \mid \theta_1^{(s)})}}_{\text{ratio of proposal densities}}$$

Glossary:

π : posterior distribution

J : proposal density aka "transition kernel"

$\underline{\theta}$: unknown parameter vector

$\underline{\theta}^{(s)}$: current state, i.e. $\{\theta_1^{(s)}, \theta_2^{(s)}, \ldots, \theta_n^{(s)}\}$

$\underline{\theta}^*$: proposed state of the vector. In this case, we are proposing a new state for θ_1 (see proposal above) while keeping other dimensions of $\underline{\theta}$ the same.

i.e. here $\underline{\theta}^* = \{\theta_1^*, \theta_2^{(s)}, \ldots, \theta_n^{(s)}\}$

Q: How is Gibbs sampling a special case of M-H?

Answer: The full conditional posterior is our proposal distribution.

Things to know about Gibbs sampling:

#1) the acceptance probability, "r" is 1.

#2) to perform Gibbs sampling, we need to be able to sample from the full conditional posteriors:

$p(\theta_1 \mid \underline{\theta}_{-1}, y)$
$p(\theta_2 \mid \underline{\theta}_{-2}, y)$
$\vdots$
$p(\theta_n \mid \underline{\theta}_{-n}, y)$

* Thing to know about full conditional posteriors:

the full conditional posterior is always proportional to the joint posterior (rule of probability).

i.e. $\boxed{p(\theta_1 \mid \cdot) \underset{\theta_1}{\propto} p(\underline{\theta} \mid y)}$

Exercise

PROOF that $r=1$ under Gibbs sampling:

Let $J(\theta_1^{(s)} | \theta_1^{*}) \equiv p(\theta_1 | \underline{\theta}_{-1}, y)$

(We propose the new state from the full cond'l posterior.)

Then r is

$$= \underbrace{\frac{p(\theta_1^*, \theta_2^{(s)}, \dots, \theta_n^{(s)} | y)}{p(\theta_1^{(s)}, \theta_2^{(s)}, \dots, \theta_n^{(s)} | y)}}_{\text{ratio of posteriors}} \cdot \underbrace{\frac{p(\theta_1 | \underline{\theta}_{-1}, y)}{p(\theta_1^* | \underline{\theta}_{-1}, y)}}_{\text{ratio of transition kernels}}$$

$$= \frac{p(\theta_1^* \cancel{| \underline{\theta}_{-1}^{(s)}}, y)\, \cancel{p(\underline{\theta}_{-1}^{(s)} | y)}}{p(\theta_1^{(s)} \cancel{| \underline{\theta}_{-1}^{(s)}}, y)\, \cancel{p(\underline{\theta}_{-1}^{(s)} | y)}} \cdot \frac{p(\theta_1^{(s)} \cancel{| \underline{\theta}_{-1}}, y)}{p(\theta_1^* \cancel{| \underline{\theta}_{-1}}, y)} = 1$$

$\square$