ESTIMATORS

new

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Bias Example #1

Let $Y_i | \theta, \sigma^2 \overset{iid}{\sim} \text{Normal}(\theta, \sigma^2)$ σ^2 fixed, known

Let $\hat{\theta}_e = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

$$\text{Bias}(\hat{\theta}_e | \theta = \theta_0) = \mathbb{E}[\hat{\theta}_e | \theta_0] - \theta_0$$
$$= \frac{1}{n} \sum \mathbb{E}[Y_i | \theta_0] - \theta_0$$
$$= \frac{1}{n} \cdot n\theta_0 - \theta_0$$
$$= 0$$

The sample mean $\hat{\theta}_e$ is an unbiased estimator of θ.

Bias Example #2

Let $X_i \overset{iid}{\sim} \text{exponential}(\theta)$

Let $\hat{\theta} = \frac{1}{\bar{x}}$ where $\bar{x} = \frac{1}{n} \sum x_i$

Is $\hat{\theta}$ biased?

$$\mathbb{E}[\hat{\theta} | \theta = \theta_0] = \mathbb{E}\left[\frac{1}{\bar{x}} | \theta_0\right] = \mathbb{E}\left[\frac{n}{\sum x_i} | \theta_0\right]$$

Notice $\sum x_i | \theta \sim \text{gamma}(n, \theta)$ (sum of independent exp w/ identical rate)

Let $y = \sum x_i \Rightarrow y | \theta \sim \text{gamma}(n, \theta)$

$\frac{1}{y} | \theta \sim \text{inv-gamma}(n, \theta)$

$$\boxed{n \cdot \mathbb{E}\left[\frac{1}{y} | \theta_0\right] = \frac{\theta_0}{n-1} \cdot n}$$

$\hat{\theta} = \frac{1}{\bar{x}}$ is a biased estimator of θ.

VARIANCE EXAMPLE

AGAIN $Y_i|\theta,\sigma^2 \overset{iid}{\sim} \text{normal}(\theta,\sigma^2)$; $\hat{\theta}_e = \bar{y}$

$$\text{var}(\hat{\theta}_e|\theta_0) = \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(y_i|\theta_0)$$

$$= \boxed{\frac{\sigma^2}{n}}$$

MSE EXAMPLE:

$$\text{MSE}(\hat{\theta}_e|\theta_0) = \underbrace{\frac{\sigma^2}{n}}_{\text{variance}} + \underbrace{0^2}_{\text{bias}^2}$$

EXERCISE #3

Let $\hat{\theta}_b = w\bar{y} + (1-w)\mu_0$

$$\text{MSE}(\hat{\theta}_b|\theta_0) = E[(\hat{\theta}_b - \theta_0)^2|\theta_0]$$

WE COULD COMPUTE MSE = VARIANCE + BIAS², BUT HERE IS A TRICK TO COMPUTE DIRECTLY:

TRICK: $(\hat{\theta}_b - \theta_0)^2 = (\hat{\theta}_b - \underbrace{(w\theta_0 + (1-w)\theta_0)}_{\theta_0})^2$

So we have: Plug in $\hat{\theta}_b$ & collect terms:

$$\mathbb{E}[(\hat{\theta}_b - (w\theta_0 + (1-w)\theta_0))^2|\theta_0].$$

$$= \mathbb{E}[(w(\bar{y}-\theta_0) + (1-w)(\mu_0-\theta_0))^2|\theta_0]$$

$$= \mathbb{E}\, w^2(\bar{y}-\theta_0)^2 + 2w(1-w)(\mu_0-\theta_0)\underbrace{\mathbb{E}(\bar{y}-\theta_0)}_{0} + \underbrace{\mathbb{E}(1-w)^2(\mu_0-\theta_0)^2}_{\text{constant}}$$

EXERCISE #3 CONTINUED

$$= w^2 \underbrace{\mathbb{E}(\bar{y} - \theta_0)^2 | \theta_0}_{var(\bar{y} | \theta_0)} + (1-w)^2 (\mu_0 - \theta_0)^2$$

$MSE(\hat{\theta}_b | \theta_0) < MSE(\hat{\theta}_e | \theta_0)$ if

$$w^2 \, var(\bar{y} | \theta_0) + (1-w)^2 (\mu_0 - \theta_0)^2 < var(\bar{y} | \theta_0)$$

Rearranging:

$$(\mu_0 - \theta_0)^2 < \frac{1+w}{1-w} var(\bar{y} | \theta_0)$$

i.e. if $\uparrow$ is small enough.

In words: If our prior guess μ_0 is "close" to θ_0, then our Bayesian estimator will have smaller MSE.